Math 60 Appendix C.2 – 2nd 3×3 Systems

Objectives

1) Solve an inconsistent 3×3 system (no solution)

2) Solve a consistent dependent 3×3 system (set of infinitely many solutions)

① Solve
$$\begin{cases} 2x - 2y + 3z = 6 & \text{(A)} \\ 4x - 3y + 2z = 0 & \text{(B)} \\ -2x + 3y - 7z = 1 & \text{(C)} \end{cases}$$

Note: We don't (and can't) know the system is inconsistent or consistent dependent by looking.
We must solve as before

* Must use all 3 equations 2 at a time
* Must eliminate the same variable twice

options:

$\boxed{\begin{array}{l} \text{elim } x \\ \begin{cases} \text{(A)} \\ \text{(C)} \end{cases} \\ \begin{cases} \text{(B)} \\ \text{(C)} \times 2 \end{cases} \end{array}}$

elim y

$\begin{cases} \text{(B)} \\ \text{(C)} \end{cases}$

$\begin{cases} \text{(A)} \times 3 \\ \text{(C)} \times 2 \end{cases}$

elim z

yuck. ☹

⇓

$$\begin{array}{rcl} 2x - 2y + 3z = 6 & & \text{(A)} \\ -2x + 3y - 7z = 1 & & \text{(C)} \\ \hline y - 4z = 7 & & \text{(D)} \end{array}$$

$$\begin{array}{rcl} 4x - 3y + 2z = 0 & & \text{(B)} \\ -4x + 6y - 14z = 2 & & \text{(C)} \times 2 \\ \hline 3y - 12z = 2 & & \text{(E)} \end{array}$$

$$\begin{cases} \text{(D)} & y - 4z = 7 \\ \text{(E)} & 3y - 12z = 2 \end{cases}$$

$$\begin{array}{rcl} 3y - 12z = 2 & & \text{(E)} \\ -3y + 12z = -21 & & \text{(D)} \times (-3) \\ \hline 0 \neq -19 & & \end{array}$$

All variables gone + False statement

Solve: $\boxed{\text{no solution}}$

Classify: $\boxed{\text{inconsistent}}$

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② Solve $\begin{cases} x - y - z = 1 & (A) \\ -x + 2y - 3z = -4 & (B) \\ 3x - 2y - 7z = 0 & (C) \end{cases}$

options:

elim x

$\begin{cases} (A) \\ (B) \end{cases}$

$\begin{cases} (C) \\ (B) \times 3 \end{cases}$

elim y

$\begin{cases} (B) \\ (C) \end{cases}$

$\begin{cases} (A) \times 2 \\ (B) \end{cases}$

elim z

yuck. ☹

Both elim x and elim y are good choices.

elim y:

$$\begin{array}{lrcl} (B) & -x + 2y - 3z & = & -4 \\ (C) & 3x - 2y - 7z & = & 0 \\ \hline & 2x \quad\quad - 10z & = & -4 \quad (D) \end{array}$$

$$\begin{array}{lrcl} 2\times(A) & 2x - 2y - 2z & = & 2 \\ (B) & -x + 2y - 3z & = & -4 \\ \hline & x \quad\quad - 5z & = & -2 \quad (E) \end{array}$$

$$\begin{cases} (D) & 2x - 10z = -4 \\ (E) & x - 5z = -2 \end{cases}$$

$$\begin{array}{lrcl} (D) & 2x - 10z & = & -4 \\ (-2)\times(E) & -2x + 10z & = & 4 \\ \hline & 0 \quad\quad 0 & = & 0 \end{array}$$

All variables gone + true statement

Set of infinitely many solutions has this structure

$\{(x,y,z) \mid x = \underbrace{\sim\sim}_{\text{an expression using } z \text{ only}},\ y = \underbrace{\sim\sim}_{\text{an expression using } z \text{ only}},\ z \text{ is a real } \# \}$

This is called a parametric solution.

Z is called the parameter. [illegible]ing z value, we can find x & y.

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To fill in the $\sim\sim\sim$ parts of the set, we look back at the work we've already done.

To get $x = \sim\sim\sim$

We need an equation containing x and z (but no y).

Either D or E will work.

E $\quad x - 5z = -2$

Isolate x :

$$x = 5z - 2$$

Our set now looks like this:

$\{(x,y,z) \mid x = 5z - 2,\ y = \sim\sim\sim,\ z \text{ is a real \#}\}$

To get $y = \sim\sim\sim$

We need an equation containing y and z (but no x).

We don't currently have one.

elim x

$$\begin{array}{lrcl} \text{A} & x - y - z &=& 1 \\ \text{B} & -x + 2y - 3z &=& -4 \\ \hline & y - 4z &=& -3 \end{array}$$

Isolate y: $\quad y = 4z - 3$

Solution set is now:

$$\boxed{\{(x,y,z) \mid \underbrace{x = 5z-2,\ y = 4z-3}_{\text{equations to get } x \,\&\, y},\ \underbrace{z \text{ is a real \#}}_{z \text{ can be any real \#}}\}}$$

set starts

set contains ordered triples

such that

set ends

Speak this: "The set of all ordered triples (x,y,z) such that $x = 5z-2$ and $y = 4z-3$, where z is any real number."

M60 App C.2 – 2nd

(3) How does

$\{(x,y,z) \mid x=5z-2,\ y=4z-3,\ z \text{ is a real \#}\}$

mean "infinitely many solutions"?

Choose any value of z

I choose $z=6$.

$x = 5(6)-2 = 28$

$y = 4(6)-3 = 21$

This gives an ordered triple $(28, 21, 6)$.

This can be repeated for all values of z.

But does it work?

Plug in and determine if $(28, 21, 6)$ is a solution.

$$\begin{cases} x-y-z=1 & \text{(A)} \\ -x+2y-3z=-4 & \text{(B)} \\ 3x-2y-7z=0 & \text{(C)} \end{cases}$$

(A) $28-21-6=1$ ✓

(B) $-28+2(21)-3(6) \stackrel{?}{=} -4$

$-28+42-18=-4$ ✓

(C) $3(28)-2(21)-7(6) \stackrel{?}{=} 0$

$84-42-42=0$ ✓

$\boxed{\text{yes!}}$

M60 App C.2 — 2nd

(4) Solve $\begin{cases} 2x - y = 2 & (A) \\ -x + 5z = 3 & (B) \\ -y + 10z = 8 & (C) \end{cases}$

options:

$\boxed{\begin{array}{l} \text{elim } x \\ \left\{\begin{array}{l} (A) \\ (B) \times 2 \end{array}\right\} \Rightarrow y \,\&\, z \ (D) \\ (C) \text{ has } y \,\&\, z \text{ only!} \\ \left\{\begin{array}{l} (C) \\ (D) \end{array}\right. \end{array}}$

(we can skip the 2nd step!)

elim y

$\left\{\begin{array}{l} (A) \ \times(-1) \Rightarrow x \,\&\, z \ (D) \\ (C) \end{array}\right.$

(B) has x & z only!

$\left\{\begin{array}{l} (B) \\ (D) \end{array}\right.$

elim z

$\left.\begin{array}{l} (B) \times(-2) \\ (C) \end{array}\right\} (D)$

(A) has $x + y$ only

$\left\{\begin{array}{l} (A) \\ (D) \end{array}\right.$

Any of the three options is a good choice, though elim x avoids multiplying by a negative.

$$\begin{array}{rrrrrl} (A) & 2x & -y & & = 2 & \\ 2\times (B) & -2x & & +10z & = 6 & \\ \hline & 0 & -y & +10z & = 8 & (D) \end{array}$$

$$\begin{array}{l} (C) \\ (D) \end{array} \begin{cases} -y + 10z = 8 \\ -y + 10z = 8 \end{cases}$$

These are the same line!

$0 = 0$

All variables gone + true statement

$\{(x, y, z) \mid x = \qquad , y = \qquad , z \text{ is a real \#}\}$

(B) has x & z but no y. Isolate x.

$-x + 5z = 3$

$-x = -5z + 3$

$x = 5z - 3$

$\{(x, y, z) \mid x = 5z - 3, \ y = \qquad , z \text{ is a real \#})$

M60 App C.2 – 2nd

(G) has y & z but no x $\quad -y+10z=8$

Isolate y $\quad -y=-10z+8$

$y=10z-8$

$$\boxed{\{(x,y,z) \mid x=5z-3,\ y=10z-8,\ z \text{ is a real } \#\}}$$

M60 App C.2 – 2nd

(5)
$$\begin{cases} x \qquad\quad - 3z = -3 & (A) \\ \quad 3y + 4z = -5 & (B) \\ 3x - 2y \qquad = 6 & (C) \end{cases}$$

options

$\boxed{\text{elim } x \quad \left\{\begin{matrix} (A) \times (-3) \\ (C) \end{matrix}\right\} \to y \,\&\, z \; (D) \qquad \left\{\begin{matrix} (B) \\ (D) \end{matrix}\right. \; y \,\&\, z}$

elim y
$\left\{\begin{matrix} (B) \times 2 \\ (C) \times 3 \end{matrix}\right. \Rightarrow x \,\&\, z \; (D)$
$\left\{\begin{matrix} (A) \\ (D) \end{matrix}\right. \; x \,\&\, z$

elim z
$\left\{\begin{matrix} (A) \times 4 \\ (B) \times 3 \end{matrix}\right. \Rightarrow x \,\&\, y$
$\left\{\begin{matrix} (C) \\ (D) \end{matrix}\right. \; x \,\&\, y$

$$\begin{array}{rrrrl} (-3) \times (A) & -3x & & +9z = & 9 \\ (C) & 3x & -2y & = & 6 \\ \hline & & -2y & +9z = & 15 \quad (D) \end{array}$$

Must use (B) — can pair with (D) and skip the second elim.

$$\begin{cases} (B) & 3y + 4z = -5 \\ (D) & -2y + 9z = 15 \end{cases}$$

$$\begin{array}{rrrl} 2 \times (B) & 6y & + 8z = & -10 \\ 3 \times (D) & -6y & + 27z = & 45 \\ \hline & & 35z = & 35 \\ & & z = & 1 \end{array}$$

We will get an ordered triple!

Plug ⇒ (B)
$$\begin{aligned} 3y + 4(1) &= -5 \\ 3y + 4 &= -5 \\ 3y &= -9 \\ y &= -3 \end{aligned}$$

Plug ⇒ (A)
$$\begin{aligned} x - 3(1) &= -3 \\ x - 3 &= -3 \\ x &= 0 \end{aligned}$$

Solution $\boxed{(0, -3, 1)}$

M60 App C.2 – 2nd

Extra practice

(6) Solve $\begin{cases} x + 2y + z = 4 & (A) \\ 4x + y - 3z = -2 & (B) \\ 6x + 5y - z = 6 & (C) \end{cases}$

options

elim x

$\begin{cases} (A) \times (-4) \\ (B) \end{cases}$

$\begin{cases} (C) \\ (A) \times (-6) \end{cases}$

elim y

$\begin{cases} (A) \\ (B) \times (-2) \end{cases}$

$\begin{cases} (C) \\ (B) \times (-5) \end{cases}$

elim z

$\boxed{\begin{cases} (A) \\ (C) \end{cases} \quad \begin{cases} (B) \\ (A) \times 3 \end{cases}}$

$\Downarrow$

$$\begin{array}{lr} (A) & x + 2y + z = 4 \\ (C) & 6x + 5y - z = 6 \\ \hline & 7x + 7y \quad = 10 \quad (D) \end{array}$$

$$\begin{array}{lr} (B) & 4x + y - 3z = -2 \\ 3 \times (A) & 3x + 6y + 3z = 12 \\ \hline & 7x + 7y \quad = 10 \quad (E) \end{array}$$

$$\begin{array}{lr} (D) & 7x + 7y = 10 \\ (E) & 7x + 7y = 10 \\ \hline & 0 = 0 \end{array}$$

same line $\rightarrow$ infinitely many solutions.

solution set

$\{(x, y, z) \mid x = \qquad , y = \qquad , z \text{ is a real \#}\}$

For $x =$

we need an equation with x & z but no y.

And we don't have one.

$$\begin{array}{lr} (A) & x + 2y + z = 4 \\ (-2) \times (B) & -8x - 2y + 6z = 4 \\ \hline & -7x \quad + 7z = 8 \end{array}$$

Isolate x

M60 App C.2 - 2nd

$$-7x + 7z = 8$$

$$\frac{-7x}{-7} = \frac{-7z}{-7} + \frac{8}{-7}$$

$$x = z - \frac{8}{7}$$

$\{(x, y, z) \mid x = z - \frac{8}{7},\ y = \qquad\qquad ,\ z \text{ is a real \#}\}.$

For $y =$

we need an equation with y & z but no x.

And we don't have one.

$$(-4) \times (A) \qquad -4x - 8y - 4z = -16$$

$$(B) \qquad 4x + y - 3z = -2$$

$$-7y - 7z = -18$$

Isolate y

$$\frac{-7y}{-7} = \frac{7z}{-7} - \frac{18}{-7}$$

$$y = -z + \frac{18}{7}$$

$$\boxed{\{(x, y, z) \mid x = z - \frac{8}{7},\ y = -z + \frac{18}{7},\ z \text{ is a real \#}\}}$$

M60 App C.2 – 2nd

Extra practice

7 $\begin{cases} -x+4y-z=8 & A \\ 4x-y+3z=9 & B \\ 2x+7y+z=0 & C \end{cases}$

options

elim x

$\begin{cases} A \times 2 \\ C \end{cases}$

$\begin{cases} B \\ C \times (-2) \end{cases}$ or $\begin{cases} B \\ A \times 4 \end{cases}$

elim y

$\begin{cases} A \\ B \times 4 \end{cases}$

$\begin{cases} C \\ B \times 7 \end{cases}$

$\boxed{\text{elim } z \quad \begin{cases} A \\ C \end{cases} \quad \begin{cases} B \\ A \times 3 \end{cases}}$

$\Downarrow$

A $-x+4y-z=8$

C $2x+7y+z=0$

$x+11y \quad =8$ D

B $4x-y+3z=9$

$3 \times$ A $-3x+12y-3z=24$

$x+11y \quad = 33$ E

D $x+11y=8$

E $x+11y=33$

D $x+11y=8$

$(-1)\times$ E $-x-11y=-33$

$0 \neq -25$ all variables gone + false statement

$\boxed{\text{NO SOLUTION}}$

M60 App C.2 - 2nd

8
$$\begin{cases} x - y + 3z = 2 & (A) \\ -2x + 3y - 8z = -1 & (B) \\ 2x - 2y + 4z = 7 & (C) \end{cases}$$

options

elim x

$\begin{cases} B \\ C \end{cases}$

$\begin{cases} A \times 2 \\ B \end{cases}$

elim y

$\begin{cases} A \times (-2) \\ C \end{cases}$

$\begin{cases} B \\ A \times 3 \end{cases}$

elim z

B
C × 2

A × 4
C × (−3)

Sometimes an interesting twist can happen:

$$\begin{array}{lr} (-2) \times A & -2x + 2y - 6z = -4 \\ C & 2x - 2y + 4z = 7 \\ \hline & 0 + 0 - 2z = 3 \quad D \end{array}$$

x and y were both eliminated!!

Solve for z!
$$-2z = 3$$
$$z = \frac{-3}{2}$$

$$\begin{array}{lr} B & -2x + 3y - 8z = -1 \\ 3 \times A & 3x - 3y + 9z = 6 \\ \hline & x + z = 5 \quad E \end{array}$$

Plug in z! ⇒ E
$$x + \left(-\frac{3}{2}\right) = 5$$
$$x = \frac{5}{1} + \frac{3}{2}$$
$$x = \frac{13}{2}$$

Plug x & z to get y. ⇒ A
$$\frac{13}{2} - y + 3\left(-\frac{3}{2}\right) = 2$$
$$-y + 2 = 2$$
$$-y = 0$$
$$y = 0$$

Solution
$$\boxed{\left(\frac{13}{2}, 0, \frac{-3}{2}\right)}$$