

Math 60 Appendix C.2 - 2nd 3×3 Systems

Objectives

- 1) Solve an inconsistent 3×3 system
(no solution)
- 2) Solve a consistent dependent 3×3 system
(set of infinitely many solutions)

① Solve $\begin{cases} 2x - 2y + 3z = 6 & (A) \\ 4x - 3y + 2z = 0 & (B) \\ -2x + 3y - 7z = 1 & (C) \end{cases}$

Note: We don't (and can't) know the system is inconsistent or consistent dependent by looking.

We must solve as before

* Must use all 3 equations
2 at a time

* Must eliminate the same variable twice

options:

elim x
{ (A)
{ (C)

$\{ (B)$	elim y	elim z
{ (C)		yuck. (D)

$\{ (B) \times 3$	$\{ (A) \times 3$	
{ (C) $\times 2$	{ (C) $\times 2$	



$$\begin{array}{r} 2x - 2y + 3z = 6 \\ -2x + 3y - 7z = 1 \\ \hline y - 4z = 7 \end{array} \quad \begin{array}{l} (A) \\ (C) \\ (D) \end{array}$$

$$\begin{array}{r} 4x - 3y + 2z = 0 \\ -4x + 6y - 14z = 2 \\ \hline 3y - 12z = 2 \end{array} \quad \begin{array}{l} (B) \\ (C) \times 2 \\ (E) \end{array}$$

$$\begin{array}{r} y - 4z = 7 \\ 3y - 12z = 2 \\ \hline -3y + 12z = -21 \\ \hline 0 \neq -19 \end{array} \quad \begin{array}{l} (E) \\ (D) \times (-3) \end{array}$$

All variables
gone

+
False statement

Solve: no solution

Classify: Inconsistent

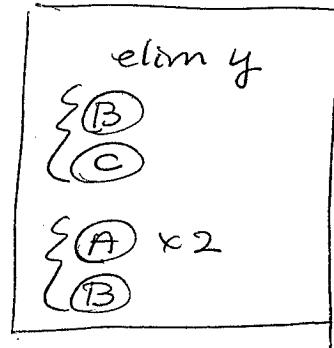
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(2) Solve $\begin{cases} x - y - z = 1 & (A) \\ -x + 2y - 3z = -4 & (B) \\ 3x - 2y - 7z = 0 & (C) \end{cases}$

options:

elim x

$$\left\{ \begin{array}{l} (A) \\ (B) \\ (C) \\ (B) \times 3 \end{array} \right.$$



Both elim x and elim y are good choices.

elim y:

$$\begin{array}{r} (B) \quad -x + 2y - 3z = -4 \\ (C) \quad 3x - 2y - 7z = 0 \\ \hline 2x \qquad \qquad -10z = -4 \end{array} \quad (D)$$

$$\begin{array}{r} 2 \times (A) \quad 2x - 2y - 2z = 2 \\ (B) \quad -x + 2y - 3z = -4 \\ \hline x \qquad \qquad -5z = -2 \end{array} \quad (E)$$

$$\left\{ \begin{array}{l} (D) \quad 2x - 10z = -4 \\ (E) \quad x - 5z = -2 \end{array} \right.$$

$$\begin{array}{r} (D) \quad 2x - 10z = -4 \\ (-2) \times (E) \quad -2x + 10z = 4 \\ \hline 0 \qquad 0 = 0 \end{array} \quad \begin{array}{l} \text{All variables} \\ \text{gone} \\ + \\ \text{true statement} \end{array}$$

Set of infinitely many solutions has this structure

$$\{(x, y, z) \mid x = \underset{\substack{\uparrow \\ \text{an expression}}}{\text{mm}}, y = \underset{\substack{\uparrow \\ \text{an expression}}}{\text{mm}}, z \text{ is a real } \#\}$$

This is called a parametric solution.
z is called the parameter. Given a value for z, we can find x & y.

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To fill in the uuu parts of the set, we look back at the work we've already done.

To get $x = uuu$

We need an equation containing x and z (but no y).

Either (D) or (E) will work.

$$(E) \quad x - 5z = -2$$

Isolate x :

$$x = 5z - 2$$

Our set now looks like this:

$$\{(x, y, z) \mid x = 5z - 2, y = uuu, z \text{ is a real } \#\}$$

To get $y = uuu$

We need an equation containing y and z (but no x).

We don't currently have one.

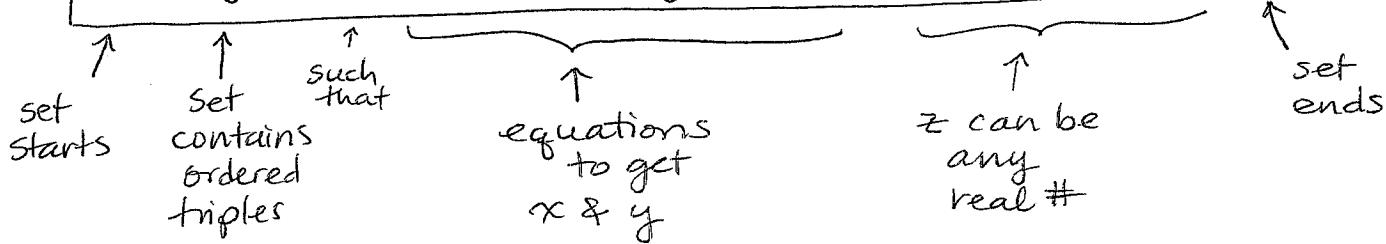
elim x

(A)	$x - y - z = 1$
(B)	$-x + 2y - 3z = -4$
	$y - 4z = -3$

Isolate y : $y = 4z - 3$

Solution set is now:

$$\boxed{\{(x, y, z) \mid x = 5z - 2, y = 4z - 3, z \text{ is a real } \#\}}$$



Speak this: "The set of all ordered triples (x, y, z) such that $x = 5z - 2$ and $y = 4z - 3$, where z is any real number."

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(3) How does

$$\{(x, y, z) \mid x = 5z - 2, y = 4z - 3, z \text{ is a real } \# \}$$

mean "infinitely many solutions"?

Choose any value of z I choose $z = 6$.

$$x = 5(6) - 2 = 28$$

$$y = 4(6) - 3 = 21$$

This gives an ordered triple $(28, 21, 6)$.This can be repeated for all values of z .

But does it work?

Plug in and determine if $(28, 21, 6)$ is a solution.

$$\begin{cases} x - y - z = 1 & \textcircled{A} \\ -x + 2y - 3z = -4 & \textcircled{B} \\ 3x - 2y - 7z = 0 & \textcircled{C} \end{cases}$$

$$\textcircled{A} \quad 28 - 21 - 6 = 1 \checkmark$$

$$\textcircled{B} \quad -28 + 2(21) - 3(6) \stackrel{?}{=} -4$$

$$-28 + 42 - 18 = -4 \checkmark$$

$$\textcircled{C} \quad 3(28) - 2(21) - 7(6) \stackrel{?}{=} 0$$

$$84 - 42 - 42 = 0 \checkmark$$

Yes!

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$$\textcircled{4} \text{ Solve } \begin{cases} 2x - y = 2 & \textcircled{A} \\ -x + 5z = 3 & \textcircled{B} \\ -y + 10z = 8 & \textcircled{C} \end{cases}$$

options:

elim x

$$\left\{ \begin{array}{l} \textcircled{A} \\ \textcircled{B} \times 2 \end{array} \right\} \Rightarrow y \& z \textcircled{D}$$

\textcircled{C} has y & z only!

$\left\{ \begin{array}{l} \textcircled{C} \\ \textcircled{D} \end{array} \right\}$

elim y

$$\left\{ \begin{array}{l} \textcircled{A} \\ \textcircled{C} \end{array} \right\} x(-1) \Rightarrow x \& z \textcircled{D}$$

\textcircled{B} has x & z only!

$$\left\{ \begin{array}{l} \textcircled{B} \\ \textcircled{D} \end{array} \right\}$$

elim z

$$\left\{ \begin{array}{l} \textcircled{B} \\ \textcircled{C} \end{array} \right\} x(-2) \textcircled{D}$$

\textcircled{A} has x & y only

$$\left\{ \begin{array}{l} \textcircled{A} \\ \textcircled{D} \end{array} \right\}$$

(we can skip the
2nd step!)

Any of the three options is a good choice, though
elim x avoids multiplying by a negative.

$$\begin{array}{rcl} \textcircled{A} & 2x - y = 2 \\ 2x \textcircled{B} & -2x + 10z = 6 \\ \hline & 0 - y + 10z = 8 & \textcircled{D} \end{array}$$

$$\begin{array}{l} \textcircled{C} \left\{ -y + 10z = 8 \\ \textcircled{D} \left\{ -y + 10z = 8 \end{array}$$

These are the
same line!

$$0 = 0$$

All variables gone
+
true statement

$$\left\{ (x, y, z) \mid x = \quad , y = \quad , z \text{ is a real #} \right\}$$

\textcircled{B} has x & z but no y
Isolate x.

$$\begin{aligned} -x + 5z &= 3 \\ -x &= -5z + 3 \\ x &= 5z - 3 \end{aligned}$$

$$\left\{ (x, y, z) \mid x = 5z - 3, y = \quad , z \text{ is a real #} \right)$$

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⑥ has y & z but no x $-y + 10z = 8$

Isolate y $-y = -10z + 8$

$$y = 10z - 8$$

$\{ z(x, y, z) \mid x = 5z - 3, y = 10z - 8, z \text{ is a real } \# \}$

M60 App C.2 - 2nd

$$\textcircled{5} \left\{ \begin{array}{l} x - 3z = -3 \\ 3y + 4z = -5 \\ 3x - 2y = 6 \end{array} \right. \quad \begin{array}{l} \textcircled{A} \\ \textcircled{B} \\ \textcircled{C} \end{array}$$

options

elim x

$$\left\{ \begin{array}{l} \textcircled{A} \times (-3) \\ \textcircled{C} \end{array} \right\} \rightarrow y \& z \textcircled{D}$$

elim y

$$\left\{ \begin{array}{l} \textcircled{B} \times 2 \\ \textcircled{C} \times 3 \end{array} \right\} \Rightarrow x \& z \textcircled{D}$$

elim z

$$\left\{ \begin{array}{l} \textcircled{A} \times 4 \\ \textcircled{B} \times 3 \end{array} \right\} \Rightarrow x \& y$$

$$\left\{ \begin{array}{l} \textcircled{B} \\ \textcircled{D} \end{array} \right\} y \& z$$

$$\left\{ \begin{array}{l} \textcircled{A} \\ \textcircled{D} \end{array} \right\} x \& z$$

$$\left\{ \begin{array}{l} \textcircled{C} \\ \textcircled{D} \end{array} \right\} x \& y$$



$$\begin{array}{r} (-3) \times \textcircled{A} \quad -3x \quad +9z = 9 \\ \textcircled{C} \quad 3x \quad -2y \quad = 6 \\ \hline -2y + 9z = 15 \end{array} \quad \textcircled{D}$$

Must use \textcircled{B} — can pair with \textcircled{D} and skip the second elim.

$$\left\{ \begin{array}{l} \textcircled{B} \\ \textcircled{D} \end{array} \right\} \begin{array}{l} 3y + 4z = -5 \\ -2y + 9z = 15 \end{array}$$

$$\begin{array}{r} 2 \times \textcircled{B} \quad 6y + 8z = -10 \\ 3 \times \textcircled{D} \quad -6y + 27z = 45 \\ \hline 35z = 35 \\ z = 1 \end{array}$$

We will get an ordered triple!

Plug $\Rightarrow \textcircled{B}$

$$\begin{aligned} 3y + 4(1) &= -5 \\ 3y + 4 &= -5 \\ 3y &= -9 \\ y &= -3 \end{aligned}$$

Plug $\Rightarrow \textcircled{A}$

$$\begin{aligned} x - 3(1) &= -3 \\ x - 3 &= -3 \\ x &= 0 \end{aligned}$$

Solution

$$(0, -3, 1)$$

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Extra practice

⑥ Solve $\begin{cases} x + 2y + z = 4 & \textcircled{A} \\ 4x + y - 3z = -2 & \textcircled{B} \\ 6x + 5y - z = 6 & \textcircled{C} \end{cases}$

options

elim x

$$\begin{cases} \textcircled{A} \times (-4) \\ \textcircled{B} \end{cases}$$

$$\begin{cases} \textcircled{C} \\ \textcircled{A} \times (-6) \end{cases}$$

elim y

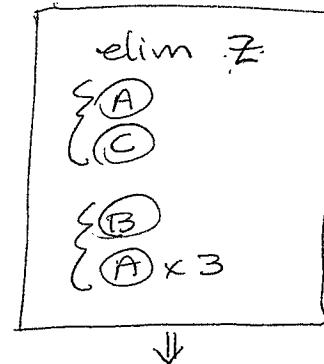
$$\begin{cases} \textcircled{A} \\ \textcircled{B} \times (-2) \end{cases}$$

$$\begin{cases} \textcircled{C} \\ \textcircled{B} \times (-5) \end{cases}$$

elim z

$$\begin{cases} \textcircled{A} \\ \textcircled{C} \end{cases}$$

$$\begin{cases} \textcircled{B} \\ \textcircled{A} \times 3 \end{cases}$$



$$\textcircled{A} \quad x + 2y + z = 4$$

$$\textcircled{C} \quad 6x + 5y - z = 6$$

$$\underline{7x + 7y = 10} \quad \textcircled{D}$$

$$\textcircled{B} \quad 4x + y - 3z = -2$$

$$3 \times \textcircled{A} \quad 3x + 6y + 3z = 12$$

$$\underline{7x + 7y = 10} \quad \textcircled{E}$$

$$\textcircled{D} \quad 7x + 7y = 10$$

$$\textcircled{E} \quad \underline{7x + 7y = 10}$$

$$0 = 0$$

Same line \rightarrow infinitely many solutions.

Solution set

$$\{(x, y, z) \mid x = . \quad y = . \quad , z \text{ is a real #}\}$$

For $x =$

We need an equation with x & z but no y .
And we don't have one.

$$\textcircled{A} \quad x + 2y + z = 4$$

$$(-2)x \textcircled{B} \quad -8x - 2y + 6z = 4$$

$$\underline{-7x + 7z = 8}$$

Isolate x

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$$-7x + 7z = 8$$

$$\frac{-7x}{-7} = \frac{-7z + 8}{-7}$$

$$x = z - \frac{8}{7}$$

$$\{(x, y, z) \mid x = z - \frac{8}{7}, y = \text{, } z \text{ is a real \#}\}$$

For $y =$

we need an equation with y & z but no x .
And we don't have one.

$$\begin{array}{rcl} (-4) \times A & -4x - 8y - 4z = -16 \\ B & \underline{4x + y - 3z = -2} \\ & -7y - 7z = -18 \end{array}$$

Isolate y

$$\frac{-7y}{-7} = \frac{7z}{-7} - \frac{18}{-7}$$

$$y = -z + \frac{18}{7}$$

$$\boxed{\{(x, y, z) \mid x = z - \frac{8}{7}, y = -z + \frac{18}{7}, z \text{ is a real \#}\}}$$

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Extra practice

$$\textcircled{7} \quad \begin{cases} -x + 4y - z = 8 & \textcircled{A} \\ 4x - y + 3z = 9 & \textcircled{B} \\ 2x + 7y + z = 0 & \textcircled{C} \end{cases}$$

options

elim x

$$\begin{cases} \textcircled{A} \times 2 \\ \textcircled{C} \end{cases}$$

$$\begin{cases} \textcircled{B} \\ \textcircled{C} \times (-2) \end{cases} \text{ or } \begin{cases} \textcircled{B} \\ \textcircled{A} \times 4 \end{cases}$$

elim y

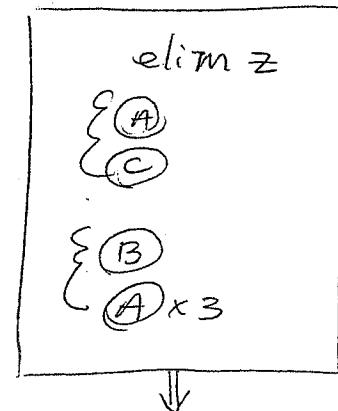
$$\begin{cases} \textcircled{A} \\ \textcircled{B} \times 4 \end{cases}$$

$$\begin{cases} \textcircled{C} \\ \textcircled{B} \times 7 \end{cases}$$

elim z

$$\begin{cases} \textcircled{A} \\ \textcircled{C} \end{cases}$$

$$\begin{cases} \textcircled{B} \\ \textcircled{A} \times 3 \end{cases}$$



$$\textcircled{A} \quad -x + 4y - z = 8$$

$$\textcircled{C} \quad 2x + 7y + z = 0$$

$$\underline{-x + 11y = 8 \quad \textcircled{D}}$$

$$\textcircled{B} \quad 4x - y + 3z = 9$$

$$3x \textcircled{A} \quad -3x + 12y - 3z = 24$$

$$\underline{x + 11y = 33 \quad \textcircled{E}}$$

$$\textcircled{D} \quad x + 11y = 8$$

$$\textcircled{E} \quad x + 11y = 33$$

$$\textcircled{D} \quad x + 11y = 8$$

$$(-1)x \textcircled{E} \quad \underline{-x - 11y = -33} \quad \begin{matrix} \text{all variables gone} \\ + \\ \text{false statement} \end{matrix}$$

NO SOLUTION

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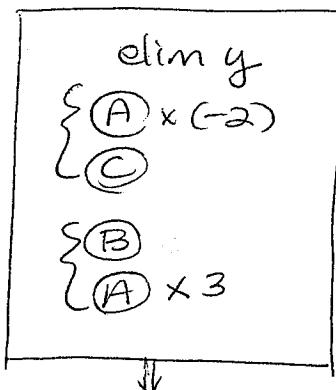
$$\textcircled{8} \left\{ \begin{array}{l} x - y + 3z = 2 \quad \textcircled{A} \\ -2x + 3y - 8z = -1 \quad \textcircled{B} \\ 2x - 2y + 4z = 7 \quad \textcircled{C} \end{array} \right.$$

options

elim x

$$\left\{ \begin{array}{l} \textcircled{B} \\ \textcircled{C} \end{array} \right.$$

$$\left\{ \begin{array}{l} \textcircled{A} \times 2 \\ \textcircled{B} \end{array} \right.$$



elim z

$$\left\{ \begin{array}{l} \textcircled{B} \\ \textcircled{C} \times 2 \end{array} \right.$$

$$\left\{ \begin{array}{l} \textcircled{A} \times 4 \\ \textcircled{C} \times (-3) \end{array} \right.$$

Sometimes an interesting twist can happen:

$$\begin{array}{rcl} (-2) \times \textcircled{A} & -2x + 2y - 6z = -4 \\ \textcircled{C} & 2x - 2y + 4z = 7 \\ \hline & 0 + 0 - 2z = 3 & \textcircled{D} \end{array}$$

$$\begin{array}{rcl} \text{Solve for } z! & -2z = 3 \\ & z = -\frac{3}{2} \end{array}$$

x and y were both eliminated!!

$$\begin{array}{rcl} \textcircled{B} & -2x + 3y - 8z = -1 \\ 3 \times \textcircled{A} & 3x - 3y + 9z = 6 \\ \hline & x + z = 5 & \textcircled{E} \end{array}$$

$$\begin{array}{rcl} \text{Plug in } z! \Rightarrow \textcircled{E} & x + (-\frac{3}{2}) = 5 \\ & x = \frac{5}{1} + \frac{3}{2} \end{array}$$

$$x = \frac{13}{2}$$

Plug x & z to get y. $\Rightarrow \textcircled{A}$

$$\begin{array}{rcl} \frac{13}{2} - y + 3(-\frac{3}{2}) & = 2 \\ -y + 2 & = 2 \\ -y & = 0 \\ y & = 0 \end{array}$$

Solution

$$\left(\frac{13}{2}, 0, -\frac{3}{2} \right)$$